## 4.5 Relative Motion

### Relative Motion

#### Key Ideas

- When analyzing the motion of an object, the reference frame in terms of position, velocity, and acceleration needs to be specified.
- If two reference frames are moving relative to each other at a constant velocity, then the accelerations of an object as observed in both reference frames are equal.

#### Learning Objectives

**After completing this section, you should be able to...**

- explain the concept of reference frames,
- write the position and velocity vector equations for relative motion,
- draw the position and velocity vectors for relative motion, and
- analyze one-dimensional and two-dimensional relative motion problems using the position and velocity vector equations.

Motion does not happen in isolation. If you're riding in a train moving at $10\ \mathrm{m/s}$ toward the east, this velocity is measured relative to the ground on which you're traveling. However, if another train passes you moving at $15\ \mathrm{m/s}$ also moving toward the east on a neighboring track, your velocity relative to the other train is different from your velocity relative to the ground. Your velocity relative to the other train is actually $5\ \mathrm{m/s}$ toward the west.

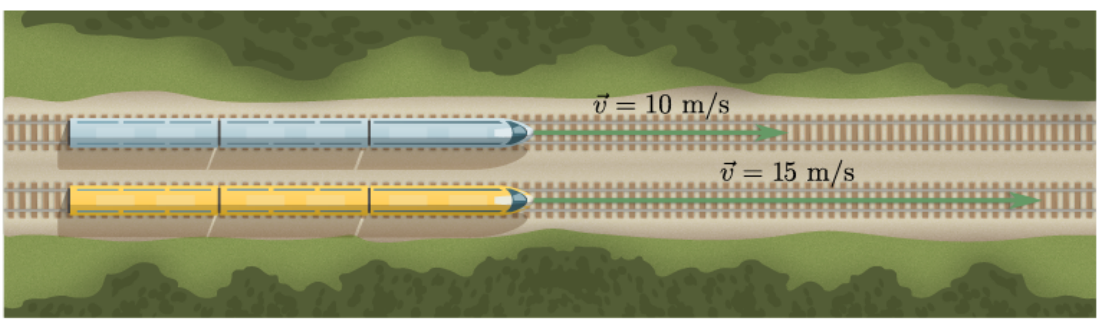



**Figure 4.26** A woman on the slower moving train sees the faster moving train moving away from her at $+5\ \mathrm{m/s}$. Equivalently, if you were a passenger at the slower train, you might view the other train to be at rest and you were moving toward the west with a velocity of $-5\ \mathrm{m/s}$.

To explore this idea further, we first need to establish some terminology.

### Reference Frames

To discuss relative motion in one or more dimensions, we first introduce the concept of reference frames. Just as a picture frame presents a context within which to see a picture, a **reference frame** provides a reference point as context to motion we are describing. When we say an object has a certain velocity, we must state it has a velocity with respect to a particular reference frame. If you say a person is sitting in a train moving at $10\ \mathrm{m/s}$, toward the east, then you imply the person on the train is moving relative to the surface of Earth at this velocity: Earth is the point of reference for describing the motion, so we say that Earth is the reference frame. In most examples we have examined so far, this reference frame has been Earth. We could expand our view of the motion of the person on the train and say Earth is spinning in its orbit around the Sun, in which case the motion becomes a lot more complicated. In this case, the solar system becomes the reference frame. In summary, all discussion of relative motion must define the reference frames involved. We now develop a method to refer to reference frames in relative motion. This method makes an approximation that breaks down as speeds become a significant fraction of the speed of light, where more accurate methods of special relativity theory are needed, but the method below is extremely accurate for everyday speeds. As needed, we'll attach a Cartesian coordinate system to a frame of reference in order to measure displacements, velocities, and accelerations.

### Relative Motion in One Dimension

We introduce relative motion in one dimension first, because the velocity vectors simplify to having only two possible directions. Take the example of a woman sitting in the slower eastbound train above. If we choose east as the positive direction and Earth as the reference frame, then we can write the velocity of *the train with respect to the Earth* as $\vec{v}_{TE} = +10\ m/s$, where the plus sign (+) refers to motion in the eastward direction. Let’s now say the woman gets up out of her seat and walks toward the back of the train at $2\ m/s$ with respect to the train. This tells us she has a velocity relative to the reference frame of the train. Since the woman is walking west, in the negative direction, we write her velocity with respect to the train as $\vec{v}_{WT} = -2 m/s$. We can add the two velocity vectors to find the velocity of the person with respect to Earth. This relative velocity is written as

$$\vec{v}_{WE} = \vec{v}_{WT} + \vec{v}_{TE} \quad (4.29)$$

To know how fast the person is moving with respect to the Earth, given how fast she is moving with respect to the train, we need to know how fast the train is moving with respect to the Earth. In this example, the train is thus a sort of intermediary coupling between the person and the Earth, and so we refer to the train as the **coupling reference frame**. Note the ordering of the subscripts for the various reference frames in Equation 4.29. The subscripts for the coupling reference frame, which is the train, appear consecutively in the right-hand side of the equation. The following figure shows the correct order of subscripts when forming the vector equation.

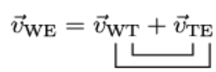



**Figure 4.27** When constructing the vector equation, the subscripts for the coupling reference frame appear consecutively on the inside. The subscripts on the left-hand side of the equation are the same as the two outside subscripts on the right-hand side of the equation.

Graphically, this is represented as follows.

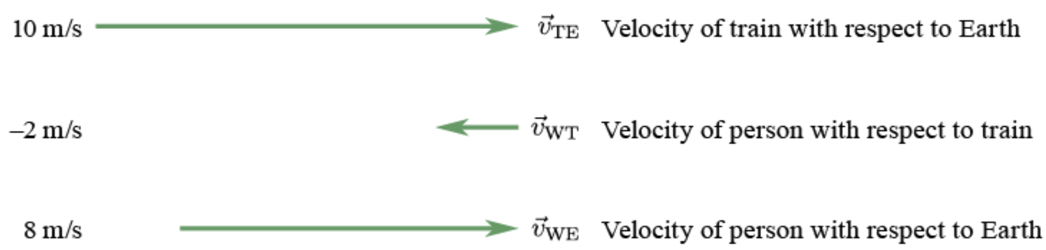
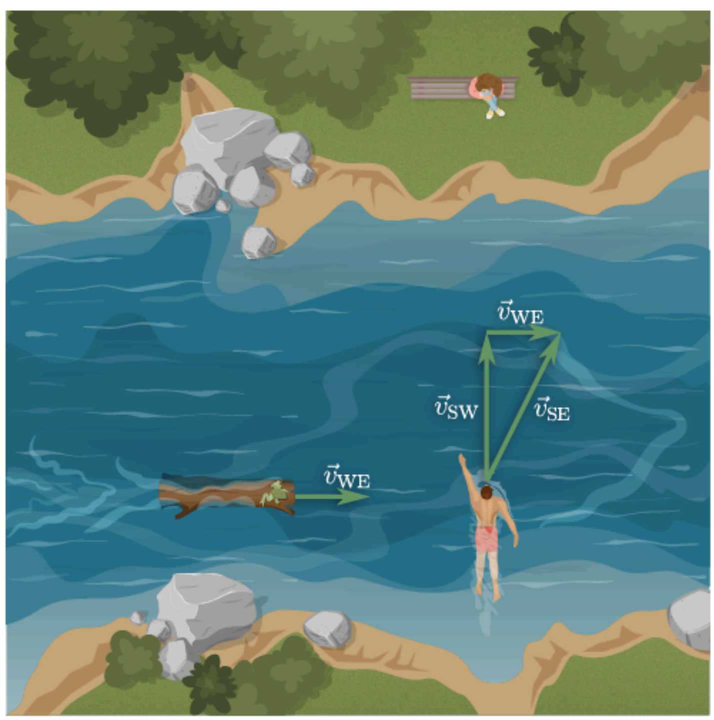



**Figure 4.28** The velocity vectors of the train with respect to Earth, the woman with respect to the train, and the woman with respect to Earth are represented.

Adding the vectors, we find $\vec{v}_{WE} = +8 m/s$, so the person is moving 8 m/s east with respect to Earth.

### Relative Motion in Two Dimensions

We can now apply these concepts to describing motion in two dimensions. Suppose a swimmer swims at a constant speed of $0.7$ m/s across a river, perpendicular to the river current. Suppose a frog sitting on a log moving at the speed of the river. From the frog's perspective, the swimmer is moving at $0.7$ m/s. That is, in the reference frame of the flowing water, the swimmer's speed is $v_{SW} = 0.7$ m/s, where the subscript is read as "the speed of the swimmer with respect to the water." Since the water is flowing, the observer sitting on the river bank will see the swimmer travel faster than this. Note that the swimmer's velocity is perpendicular to the velocity of the river current. If the river flows at $0.3$ m/s relative to the stationary bank (Earth frame), we write $v_{WE} = 0.3 m/s$, then the person sitting on the river bank measures the velocity of the swimmer as follows.




The velocity of the swimmer relative to the water is perpendicular to the velocity of the water relative to the earth. The magnitude of the relative velocity of the swimmer with respect to the Earth is found using the Pythagorean theorem.

$$\begin{aligned}\vec{v}_{SE} &= \vec{v}_{SW} + \vec{v}_{WE} \\ v_{SE} &= \sqrt{v_{SW}^2 + v_{WE}^2} \\ &= \sqrt{(0.7\ \text{m/s})^2 + (0.3\ \text{m/s})^2} \\ &= 0.8\ \text{m/s}\end{aligned}$$

Let us generalize the example above. We identify two frames that move relative to each other (the river bank, or Earth, and the moving water in this example) as reference frames $\mathbf{S}$ and $\mathbf{S'}$. We identify an origin with Cartesian axes in each reference frame that is stationary in its reference frame (the person on the bank and the frog on the log, respectively). We want to talk about the position, velocity, and acceleration of a particle $\mathbf{P}$ (the swimmer) in each of these two reference frames as shown in the following figure. The position of the origin of frame $\mathbf{S'}$ as measured in frame $\mathbf{S}$ is $\vec{\mathbf{r}}_{\mathbf{S'S}}$ and the position of $\mathbf{P}$ as measured in frame $\mathbf{S'}$ is $\vec{\mathbf{r}}_{\mathbf{PS'}}$ and the position of $\mathbf{P}$ as measured in frame $\mathbf{S}$ is $\vec{\mathbf{r}}_{\mathbf{PS}}$.

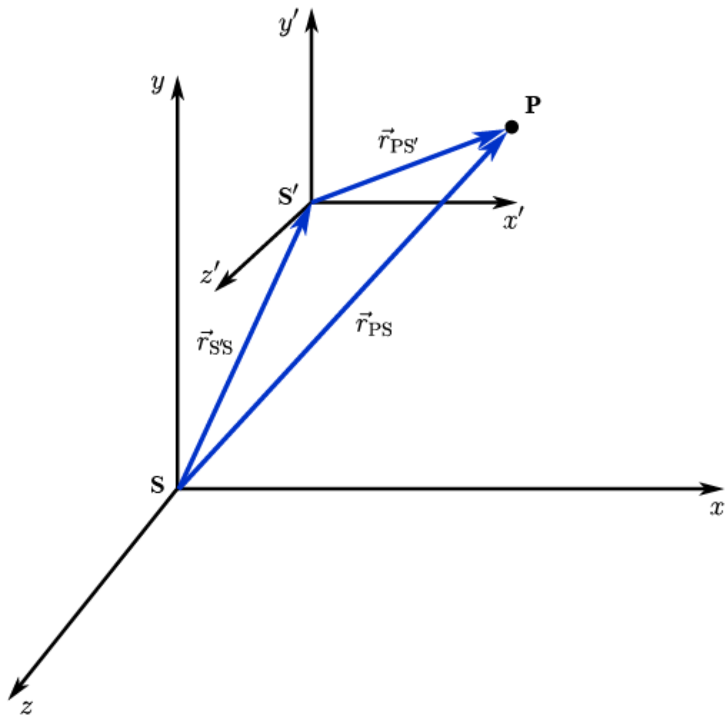



**Figure 4.29** The positions of particle $\mathbf{P}$ relative to frames $\mathbf{S}$ and $\mathbf{S'}$ are illustrated.

That is, in the reference frame of the river bank, the swimmer's speed is $0.8\ \text{m/s}$. In this example, the moving water is the coupling reference frame, as it couples the swimmer's velocity relative to the water with its velocity relative to the Earth. The swimmer cannot swim perpendicular to the water current and arrive at a point directly across from his starting point. Instead, the river current carries the swimmer downstream some distance.

$$\vec{\mathbf{r}}_{\mathbf{PS}} = \vec{\mathbf{r}}_{\mathbf{PS'}} + \vec{\mathbf{r}}_{\mathbf{S'S}} \tag{4.30}$$

Relative velocities are the time derivatives of the position vectors.